

NON-LOCAL HEAT FLOWS AND GRADIENT ESTIMATES ON CLOSED MANIFOLDS

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ABSTRACT. In this paper, we study two kind of L^2 norm preserved non-local heat flows on closed manifolds. We first study the global existence, stability and asymptotic behavior to such non-local heat flows. Next we give the gradient estimates of positive solutions to these heat flows.

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1. INTRODUCTION

In this paper, we consider the global existence, stability, asymptotic behavior and gradient estimates for two kind of L^2 preserving heat flow which have positive solutions on closed manifolds. This is a continuation of our earlier study of non-local heat flows in [9] and [8]. Our work is also motivated by the recent work of C.Caffarelli and F.Lin [2], where they have studied the global existence and regularity of L^2 norm preserving heat flow such as

$$\partial_t u = \Delta u + \lambda(t)u$$

with

$$\lambda(t) = \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx}.$$

They also extend the method to study a family of singularly perturbed systems of non-local parabolic equations. We remark that the non-local heat flow naturally arises in geometry such that the flow preserves some L^p norm in the sense that some the geometrical quantity (such as length, area and so on) is preserved in the geometric heat flow. For more references on geometric flows such as harmonic map heat flows and non-local heat flows, one may see [1], [16], [8] and [9].

We firstly study the following linear heat flow on a closed Riemannian manifold M ,

$$\begin{cases} \partial_t u = \Delta u + \lambda(t)u + A(x, t) & \text{in } M \times \mathbb{R}_+, \\ u(x, 0) = g(x) & \text{in } M, \end{cases}$$

where $g \in C^1(M)$, $A(x, t)$ is a given non-negative smooth function, and $\lambda(t)$ is chosen such that the flow preserves the L^2 the norm of the solution. In fact, a direct computation shows that

$$\frac{1}{2} \frac{d}{dt} \int_M u^2 dx = \int_M uu_t = - \int_M |\nabla u|^2 dx + \lambda(t) \int_M u^2 dx + \int_M u A dx.$$

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Thus, one has $\lambda(t) = \frac{\int_M (|\nabla u|^2 - uA) dx}{\int_M g^2 dx}$ to preserve the L^2 norm. Without loss of generality we may assume $\int_M g^2 dx = 1$. Thus we are lead to consider the following problem on the closed Riemannian manifold M

$$(1.1) \quad \begin{cases} \partial_t u = \Delta u + \lambda(t)u + A(x, t) & \text{in } M \times \mathbb{R}_+, \\ u(x, 0) = g(x) & \text{in } M, \end{cases}$$

where $\lambda(t) = \int_M (|\nabla u|^2 - uA) dx$, $g(x) \geq 0$, $\int_M g^2 dx = 1$.

We next study the following nonlinear heat flow on the closed Riemannian manifold M ,

$$\begin{cases} \partial_t u = \Delta u + \lambda(t)u - u^p & \text{in } M \times \mathbb{R}_+, \\ u(x, 0) = g(x) & \text{in } M, \end{cases}$$

where $p > 1$, which has the positive solution and preserves the L^2 the norm. Likewise,

$$\frac{1}{2} \frac{d}{dt} \int_M u^2 dx = \int_M uu_t = - \int_M |\nabla u|^2 dx + \lambda(t) \int_M u^2 dx - \int_M u^{p+1} dx.$$

Thus, one must have $\lambda(t) = \frac{\int_M (|\nabla u|^2 + u^{p+1}) dx}{\int_M g^2 dx}$ to preserve the L^2 norm. Without loss of generality we assume $\int_M g^2 dx = 1$. Then we consider the following problem on closed Riemannian manifold M

$$(1.2) \quad \begin{cases} \partial_t u = \Delta u + \lambda(t)u - u^p & \text{in } M \times \mathbb{R}_+ \\ u(x, 0) = g(x) & \text{in } M \end{cases}$$

where $p > 1$, $\lambda(t) = \int_M (|\nabla u|^2 + u^{p+1}) dx$, $g(x) \geq 0$ in M , $\int_M g^2 dx = 1$ and $g \in C^1(M)$.

Similar to the global existence results obtained in C.Caffarelli and F.Lin [2], we have following two results.

Theorem 1. *Problem (1.1) has a global non-negative solution $u(t) \in L^\infty(\mathbb{R}_+, H^1(M)) \cap L^2_{loc}(\mathbb{R}_+, H^2(M))$ if $A \in L^\infty(\mathbb{R}_+, H^1(M))$.*

Theorem 2. *Problem (1.2) has a global positive solution*

$$u(t) \in L^\infty(\mathbb{R}_+, H^1(M)) \cap L^\infty(\mathbb{R}_+, L^{p+1}(M)) \cap L^2_{loc}(\mathbb{R}_+, H^2(M)).$$

We also have the stability results for both problem (1.1) and (1.2).

Theorem 3. *Let u, v be the two non-negative solutions to problem (1.1) with initial data g_u, g_v at $t = 0$. Suppose $A \in L^\infty(\mathbb{R}_+, H^1(M))$. Then*

$$\|u - v\|_{L^2}^2 \leq \|g_u - g_v\|_{L^2}^2 \exp(C_1 t)$$

and

$$\|\nabla(u - v)\|_{L^2}^2 \leq \|\nabla(g_u - g_v)\|_{L^2}^2 \exp(C_2 t),$$

where $C_i, i = 1, 2$, are constants depending on the upper bound of $\|g_u\|_{H^1(M)}, \|g_v\|_{H^1(M)}$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$. In particular, the solution to problem (1.1) is unique.

Theorem 4. *Let u, v be the two bounded positive solutions to problem (1.2) with initial data g_u, g_v at $t = 0$, where $g_u, g_v \in H^1(M) \cap L^\infty(M)$. Then*

$$\|u - v\|_{L^2}^2 \leq \|g_u - g_v\|_{L^2}^2 \exp(C_1 t)$$

and

$$\|\nabla(u - v)\|_{L^2}^2 \leq \|\nabla(g_u - g_v)\|_{L^2}^2 \exp(C_2 t),$$

where C_i , $i = 1, 2$, are constants depending on the upper bound of $\|g_u\|_{H^1(M)}$, $\|g_v\|_{H^1(M)}$ and $\|g_u\|_{L^\infty}$, $\|g_v\|_{L^\infty}$. In particular, the solution to problem (1.2) is unique.

As the simple applications to Theorem 1 and theorem 2, we can study asymptotic behavior of $u(t)$ of problem (1.1) and problem (1.2). We have the following two corollaries.

Corollary 5. Suppose $A \in L^\infty(\mathbb{R}_+, H^1(M)) \cap L^2(\mathbb{R}_+, L^2(M))$ in Theorem 1. Let $u(t)$ be the solution to problem (1.1). Then one can take $t_i \rightarrow \infty$ such that $\lambda(t_i) \rightarrow \lambda_\infty$, $u(x, t_i) \rightarrow u_\infty(x)$ in $H^1(M)$ and u_∞ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty + A = 0$ in M with $\int_M |u_\infty|^2 dx = 1$.

Corollary 6. Suppose $u(t)$ is the positive solution to problem (1.2). Then one can take $t_i \rightarrow \infty$ such that $\lambda(t_i) \rightarrow \lambda_\infty$, $u(x, t_i) \rightarrow u_\infty(x)$ in $H^1(M)$ and $u_\infty > 0$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty - u_\infty^p = 0$ in M with $\int_M |u_\infty|^2 dx = 1$.

In the study of geometric analysis as well as other elliptic or parabolic equations, it is well known that the Harnack inequality plays a important role (see for instance [4], [13] and [15]). As showed in [15], the Harnack inequality for positive solutions is a consequence of the gradient estimates for positive solutions. We have the following gradient estimates for the type of heat equations related to problem (1.1) and (1.2).

Theorem 7. Suppose M is a closed Riemannian manifold with Ricci curvature bounded from below by $\geq -K$. Let $u > 0$ be a smooth solution to the heat equation on $M \times [0, T]$

$$(\partial_t - \Delta)u = \lambda(t)u + A(x, t).$$

Let, for $a > 1$ and $w = \log u$,

$$F = t(|\nabla w|^2 - aw_t + a(\lambda + u^{-1}A)).$$

Then there is a constant $C(u^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T)$ such that

$$\sup_{M \times [0, T]} F \leq C(|u|^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T).$$

Theorem 8. Suppose M is a closed Riemannian manifold with Ricci curvature bounded from below by $\geq -K$. Let $u > 0$ be a smooth solution to the heat equation on $M \times [0, T]$

$$(\partial_t - \Delta)u = \lambda(t)u - u^p.$$

Let, for $a > 1$ and $w = \log u$,

$$F = t(|\nabla w|^2 - aw_t + a(\lambda - u^{p-1})).$$

Then there is a constant $C(u^{p-1}, K, a, p, T)$ such that

$$\sup_{M \times [0, T]} F \leq C(|u|^{p-1}, K, a, p, T).$$

The proofs of the results above are similar to our earlier work [11].

This paper is organized as follows. In section 2 we study the global existence, stability and asymptotic behavior of solutions to the problem (1.1) and problem (1.2). In particular, we give the proofs of theorem 1 to theorem 4, corollary 5 and corollary 6. In section 3 we do the gradient estimates for problem (1.1) and problem (1.2).

2. GLOBAL EXISTENCE AND STABILITY PROPERTY

In this section we study the global existence, stability and asymptotic behavior of solutions to the problem (1.1) and problem (1.2). For global existence, we use the idea of [2] theorem 1.1 and construct a series of solutions to linear parabolic equations to converge to the solution of the non-local heat flow.

Proof of Theorem 1. Let us define a series non-negative functions $u^{(k)}$ by

$$(2.1) \quad \begin{cases} u^{(0)} = g, \lambda^{(k)}(t) = \int_M (|\nabla u^{(k)}|^2 - u^{(k)} A) dx, \\ \partial_t u^{(k+1)} = \Delta u^{(k+1)} + \lambda^{(k)}(t) u^{(k+1)} + A, \\ u^{(k+1)}(x, 0) = g(x), \end{cases}$$

which are a series of initial boundary value problems of linear parabolic systems.

To prove the convergence of the series $\{u^{(k)}\}$ constructed above, we estimate for $k \geq 0$

$$(2.2) \quad \frac{1}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx + \int_M |\nabla u^{(k+1)}|^2 dx = \lambda^{(k)}(t) \int_M |u^{(k+1)}|^2 dx + \int_M u^{(k+1)} A dx,$$

$$(2.3) \quad \frac{1}{2} \frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx + \int_M |\Delta u^{(k+1)}|^2 dx = \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx + \int_M \nabla u^{(k+1)} \cdot \nabla A dx,$$

$$(2.4) \quad \frac{1}{2} \frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx + \int_M |u_t^{(k+1)}|^2 dx = \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx + \int_M u_t^{(k+1)} A dx.$$

Since $\lambda^{(k)}(t) \leq \int_M |\nabla u^{(k)}|^2$, by (2.3), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx &\leq \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx + \int_M \nabla u^{(k+1)} \cdot \nabla A dx \\ &\leq \left(\int_M |\nabla u^{(k)}|^2 \right) \int_M |\nabla u^{(k+1)}|^2 dx + \frac{1}{2} \int_M |\nabla u^{(k+1)}|^2 dx \\ &\quad + \frac{1}{2} \int_M |\nabla A|^2 dx. \end{aligned}$$

Now we define $c_1 = \max(1, \|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}^2)$, we get

$$\frac{d}{dt} \left(\int_M |\nabla u^{(k+1)}|^2 dx + c_1 \right) \leq 2 \left(\int_M |\nabla u^{(k)}|^2 dx + c_1 \right) \left(\int_M |\nabla u^{(k+1)}|^2 dx + c_1 \right).$$

Hence

$$\int_M |\nabla u^{(k+1)}|^2 dx + c_1 \leq \left(\int_M |\nabla g|^2 dx + c_1 \right) \exp \left(2 \int_0^t \left(\int_M |\nabla u^{(k)}|^2 dx + c_1 \right) dt \right).$$

By induction, there is δ depending only on $\int_M |\nabla g|^2 dx$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$ such that

$$(2.5) \quad \int_M |\nabla u^{(k+1)}|^2 dx \leq c'_1, \quad \text{for } t \in [0, \delta], k \geq 1,$$

where c'_1 is a constant depending on $\int_M |\nabla g|^2 dx$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$. Hence $\lambda^{(k+1)} \leq \int_M |\nabla u^{(k+1)}|^2 dx \leq c'_1$. By (2.2), we have

$$\begin{aligned} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx &\leq 2\lambda^{(k)}(t) \int_M |u^{(k+1)}|^2 dx + 2 \int_M u^{(k+1)} A dx \\ &\leq (2\lambda^{(k)}(t) + 1) \int_M |u^{(k+1)}|^2 dx + \int_M |A|^2 dx. \end{aligned}$$

Hence,

$$(2.6) \quad \int_M |u^{(k+1)}|^2 dx \leq c_2, \quad \text{for } t \in [0, \delta], k \geq 1,$$

where c_2 depending on $\int_M |g|^2 dx$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$. Now integrate (2.3) with t , we get

$$\begin{aligned} \frac{1}{2} \int_M |\nabla u(t)^{(k+1)}|^2 dx - \frac{1}{2} \int_M |\nabla u(0)^{(k+1)}|^2 dx + \int_0^\delta \int_M |\Delta u^{(k+1)}|^2 dx dt \\ = \int_0^\delta \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx dt + \int_0^\delta \int_M \nabla u^{(k+1)} \cdot \nabla A dx dt. \end{aligned}$$

We have

$$\begin{aligned} \int_0^\delta \int_M |\Delta u^{(k+1)}|^2 dx dt &\leq \frac{1}{2} \int_M |\nabla g|^2 dx + \int_0^\delta \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx dt \\ &\quad + \int_0^\delta \int_M \nabla u^{(k+1)} \cdot \nabla A dx dt. \end{aligned}$$

Hence,

$$(2.7) \quad \int_0^\delta \int_M |\Delta u^{(k+1)}|^2 dx dt \leq c_3,$$

where c_3 depending on $\int_M |\nabla g|^2 dx$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$. We integrate (2.4) with t and we get

$$\begin{aligned} \frac{1}{2} \int_M |\nabla u(t)^{(k+1)}|^2 dx - \frac{1}{2} \int_M |\nabla u(0)^{(k+1)}|^2 dx + \int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt \\ = \int_0^\delta \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx dt + \int_0^\delta \int_M u_t^{(k+1)} A dx dt. \end{aligned}$$

We then have

$$\begin{aligned} \int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt &\leq \frac{1}{2} \int_M |\nabla g|^2 dx + \int_0^\delta \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx dt \\ &\quad + \int_0^\delta \int_M u_t^{(k+1)} A dx dt. \end{aligned}$$

Hence

$$(2.8) \quad \int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt \leq c_4,$$

where c_4 depending on $\int_M |\nabla g|^2 dx$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$.

By (2.5), (2.6), (2.7) and (2.8), there is a subsequence of $\{u^{(k)}\}$ (still denoted by $\{u^{(k)}\}$) and a function $u(t) \in L^\infty([0, \delta], H^1(M)) \cap L^2([0, \delta], H^2(M))$ with $\partial_t u(t) \in L^2([0, \delta], L^2(M))$ such that $u^{(k)} \rightharpoonup u$ weak* in $L^\infty([0, \delta], H^1(M))$ and weakly in

$L^2([0, \delta], H^2(M))$. Then we have $u^{(k)} \rightarrow u$ strongly in $L^2([0, \delta], H^1(M))$ and $u(t) \in C([0, \delta], L^2(M))$. Hence $\lambda^{(k)}(t) \rightarrow \lambda(t)$ strongly in $L^2([0, \delta])$. Thus, we get a local strong solution to problem (1.1). Next, starting from $t = \delta$ we can extend the local solution to $[0, 2\delta]$ in exactly the same way as above. By induction, we have a global solution to problem (1.1). \square

The proof of Theorem 2 is similar to Theorem 1 with slight modification.

Proof of Theorem 2. Let us define a series $u^{(k)}$ by

$$(2.9) \quad \begin{cases} u^{(0)} = g, \lambda^{(k)}(t) = \int_M (|\nabla u^{(k)}|^2 + (u^{(k)})^{p+1}) dx, \\ \partial_t u^{(k+1)} = \Delta u^{(k+1)} + \lambda^{(k)}(t) u^{(k+1)} - (u^{(k+1)})^p, \\ u^{(k+1)}(x, 0) = g(x), \end{cases}$$

which are a series of initial boundary value problems of linear parabolic systems. By the maximum principle, we know that $u^{(k)} > 0$.

To prove the convergence of series $\{u^{(k)}\}$ constructed above, we estimate for $k \geq 0$

$$(2.10) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx + \int_M (|\nabla u^{(k+1)}|^2 + (u^{(k+1)})^{p+1}) dx \\ &= \lambda^{(k)}(t) \int_M |u^{(k+1)}|^2 dx, \end{aligned}$$

$$(2.11) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx + \int_M |\Delta u^{(k+1)}|^2 dx + \int_M p(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 dx \\ &= \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx, \end{aligned}$$

$$(2.12) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx + \int_M |u_t^{(k+1)}|^2 dx + \frac{1}{p+1} \frac{d}{dt} \int_M (u^{(k+1)})^{p+1} dx \\ &= \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx. \end{aligned}$$

$$(2.13) \quad \begin{aligned} & \frac{1}{p+1} \frac{d}{dt} \int_M (u^{(k+1)})^{p+1} dx + \int_M p(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 dx + \int_M (u^{(k+1)})^{2p} dx \\ &= \lambda^{(k)}(t) \int_M (u^{(k+1)})^{p+1} dx. \end{aligned}$$

By (2.11) and (2.13), we have $\frac{d}{dt} \int_M |\nabla u^{(k+1)}|^2 dx \leq 2\lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx$ and $\frac{d}{dt} \int_M (u^{(k+1)})^{p+1} dx \leq (p+1)\lambda^{(k)}(t) \int_M (u^{(k+1)})^{p+1} dx$. Hence $\frac{d}{dt} \lambda^{(k+1)} \leq (p+1)\lambda^{(k)}\lambda^{(k+1)}$ and $\lambda^{(k+1)} \leq \int_M (|\nabla g|^2 + g^{p+1}) dx \cdot \exp((p+1) \int_0^t \lambda^{(k+1)} dt)$. By induction, there is δ depending only on $\int_M |\nabla g|^2 dx$ and $\int_M g^{p+1} dx$ such that

$$(2.14) \quad \lambda^{(k+1)}(t) \leq c_5, \quad \text{for } t \in [0, \delta], k \geq 1,$$

where c_5 is a constant depending on $\int_M |\nabla g|^2 dx$ and $\int_M g^{p+1} dx$. Hence

$$(2.15) \quad \int_M |\nabla u^{(k+1)}|^2 dx \leq c_5, \quad \int_M |u^{(k+1)}|^{p+1} dx \leq c_5 \quad \text{for } t \in [0, \delta], k \geq 1.$$

Now we integrate (2.11) with t and we get

$$\frac{1}{2} \int_M |\nabla u(t)^{(k+1)}|^2 dx - \frac{1}{2} \int_M |\nabla u(0)^{(k+1)}|^2 dx + \int_0^t \int_M |\Delta u^{(k+1)}|^2 dx dt$$

$$+ \int_0^\delta \int_M p(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 dx dt = \int_0^\delta \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx dt.$$

We then have

$$\int_0^\delta \int_M |\Delta u^{(k+1)}|^2 dx dt \leq \frac{1}{2} \int_M |\nabla g|^2 dx + \int_0^\delta \lambda^{(k)}(t) \int_M |\nabla u^{(k+1)}|^2 dx dt.$$

Hence

$$(2.16) \quad \int_0^\delta \int_M |\Delta u^{(k+1)}|^2 dx dt \leq c_6,$$

where c_6 depending on $\int_M |g|^2 dx$ and $\int_M |\nabla g|^2 dx$. Integrating (2.12) with t , we get

$$\begin{aligned} & \frac{1}{2} \int_M |\nabla u(t)^{(k+1)}|^2 dx - \frac{1}{2} \int_M |\nabla u(0)^{(k+1)}|^2 dx + \int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt \\ & + \frac{1}{p+1} \int_M (u^{(k+1)}(t))^{p+1} dx - \frac{1}{p+1} \int_M g^{p+1} dx = \int_0^\delta \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx dt. \end{aligned}$$

We then have

$$\int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt \leq \frac{1}{2} \int_M |\nabla g|^2 dx + \frac{1}{p+1} \int_M g^{p+1} dx + \int_0^\delta \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_M |u^{(k+1)}|^2 dx dt.$$

Hence

$$(2.17) \quad \int_0^\delta \int_M |u_t^{(k+1)}|^2 dx dt \leq c_7,$$

where c_7 depending on $\int_M |g|^2 dx$, $\int_M |\nabla g|^2 dx$ and $\int_M g^{p+1} dx$.

By (2.14), (2.15), (2.16) and (2.17), there is a subsequence of $\{u^{(k)}\}$ (still denoted by $\{u^{(k)}\}$) and a function $u(t) \in L^\infty([0, \delta], H^1(M)) \cap L^2([0, \delta], H^2(M)) \cap L^\infty([0, \delta], L^{p+1}(M))$ with $\partial_t u(t) \in L^2([0, \delta], L^2(M))$ such that $u^{(k)} \rightharpoonup u$ weak* in $L^\infty([0, \delta], H^1(M))$, weakly in $L^2([0, \delta], H^2(M))$ and weakly in $L^\infty([0, \delta], L^{p+1}(M))$. Then we have $u^{(k)} \rightarrow u$ strongly in $L^2([0, \delta], H^1(M))$ and $u(t) \in C([0, \delta], L^2(M))$. Hence $\lambda^{(k)}(t) \rightarrow \lambda(t)$ strongly in $L^2([0, \delta])$. Thus, we get a local strong solution to problem (1.2). Next, starting from $t = \delta$ we can extend the local solution to $[0, 2\delta]$ in exactly the same way as above. By induction, we have a global solution to problem (1.2). \square

Now we can study the stability for the problem (1.1) and problem (1.2).

Proof of Theorem 3. By the arguments in Theorem 1, we can take a constant C such that all $\|u\|_{L^\infty(\mathbb{R}_+, H^1(M))}$, $\|v\|_{L^\infty(\mathbb{R}_+, H^1(M))}$, $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$, $\|\lambda_u(t)\|_{L^\infty(\mathbb{R}_+)}$ and $\|\lambda_v(t)\|_{L^\infty(\mathbb{R}_+)}$ not less than C , where C is only depending on upper bound of $\|g_u\|_{H^1(M)}$, $\|g_v\|_{H^1(M)}$ and $\|A\|_{L^\infty(\mathbb{R}_+, H^1(M))}$. First we calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_M (u-v)^2 dx &= \int_M (u-v)(u_t - v_t) dx \\ &= \int_M (u-v)(\Delta(u-v) + \lambda_u(t)u - \lambda_v(t)v) dx \\ &= - \int_M |\nabla(u-v)|^2 dx + \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v) dx \end{aligned}$$

Note that

$$\begin{aligned}
& \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v)dx \\
&= (\lambda_u(t) - \lambda_v(t)) \int_M (u-v)udx + \lambda_v(t) \int_M (u-v)^2dx \\
&\leq |\lambda_u(t) - \lambda_v(t)| \left(\int_M (u-v)^2dx \right)^{\frac{1}{2}} \left(\int_M u^2dx \right)^{\frac{1}{2}} + \lambda_v(t) \int_M (u-v)^2dx \\
&\leq C|\lambda_u(t) - \lambda_v(t)| \left(\int_M (u-v)^2dx \right)^{\frac{1}{2}} + C \int_M (u-v)^2dx,
\end{aligned}$$

and

$$\begin{aligned}
|\lambda_u(t) - \lambda_v(t)| &= \left| \int_M (|\nabla u|^2 - |\nabla v|^2) - (uA - vA) dx \right| \\
&\leq \int_M |\nabla(u-v)|(|\nabla u| + |\nabla v|)dx + \left| \int_M (u-v)A dx \right| \\
&\leq \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M (|\nabla u| + |\nabla v|)^2 dx \right)^{\frac{1}{2}} \\
&\quad + \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \left(\int_M A^2 dx \right)^{\frac{1}{2}} \\
(2.18) \quad &\leq C \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} + C \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

We have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_M (u-v)^2 dx \\
&\leq - \int_M |\nabla(u-v)|^2 dx + C^2 \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \\
&\quad + C^2 \int_M (u-v)^2 dx + C \int_M (u-v)^2 dx \\
&\leq - \frac{1}{2} \int_M |\nabla(u-v)|^2 dx + \left(\frac{C^4}{2} + C^2 + C \right) \int_M (u-v)^2 dx.
\end{aligned}$$

By Gronwall inequality, we have

$$\|u-v\|_{L^2}^2 \leq \|g_u - g_v\|_{L^2}^2 \exp\left(\left(\frac{C^4}{2} + C^2 + C\right)t\right).$$

Further more,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_M |\nabla(u-v)|^2 dx &= \int_M \nabla(u-v) \cdot \nabla(u-v)_t dx \\
&= - \int_M \Delta(u-v) \cdot (u-v)_t dx \\
&= - \int_M \Delta(u-v) \cdot (\Delta(u-v) + \lambda_u(t)u - \lambda_v(t)v) dx \\
&= - \int_M (\Delta(u-v))^2 dx + \int_M \nabla(u-v) \cdot \nabla(\lambda_u(t)u - \lambda_v(t)v) dx
\end{aligned}$$

Likewise,

$$\begin{aligned}
& \int_M \nabla(u-v) \cdot \nabla(\lambda_u(t)u - \lambda_v(t)v) dx \\
&= (\lambda_u(t) - \lambda_v(t)) \int_M \nabla(u-v) \cdot \nabla u dx + \lambda_v(t) \int_M |\nabla(u-v)|^2 dx \\
&\leq |\lambda_u(t) - \lambda_v(t)| \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M |\nabla u|^2 dx \right)^{\frac{1}{2}} + \lambda_v(t) \int_M |\nabla(u-v)|^2 dx \\
&\leq C |\lambda_u(t) - \lambda_v(t)| \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} + C \int_M |\nabla(u-v)|^2 dx.
\end{aligned}$$

By Poincare inequality and (2.18), we have

$$|\lambda_u(t) - \lambda_v(t)| \leq C' \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}},$$

where C' is a constant depending on the constant of Poincare inequality and C . We take a constant of the maximum of C and C' , and still denote it C for convenience. Then we have

$$\frac{1}{2} \frac{d}{dt} \int_M |\nabla(u-v)|^2 dx \leq (C^2 + C) \int_M |\nabla(u-v)|^2 dx.$$

By Gronwall inequality, we have

$$\|\nabla(u-v)\|_{L^2}^2 \leq \|\nabla(g_u - g_v)\|_{L^2}^2 \exp((C^2 + C)t).$$

□

Proof of Theorem 4. By the arguments in Theorem 2, we can take a constant C such that all $\|u\|_{L^\infty(\mathbb{R}_+, H^1(M))}$, $\|v\|_{L^\infty(\mathbb{R}_+, H^1(M))}$, $\|u\|_{L^\infty(\mathbb{R}_+, L^\infty(M))}$, $\|v\|_{L^\infty(\mathbb{R}_+, L^\infty(M))}$, $\|\lambda_u(t)\|_{L^\infty(\mathbb{R}_+)}$ and $\|\lambda_v(t)\|_{L^\infty(\mathbb{R}_+)}$ are not less than C , where C is only depending on upper bound of $\|g_u\|_{H^1(M)}$, $\|g_v\|_{H^1(M)}$, $\|g_u\|_{L^\infty(M)}$, $\|g_v\|_{L^\infty(M)}$. We calculate

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_M (u-v)^2 dx &= \int_M (u-v)(u_t - v_t) dx \\
&= \int_M (u-v)(\Delta(u-v) + \lambda_u(t)u - \lambda_v(t)v - (u^p - v^p)) dx \\
&\leq - \int_M |\nabla(u-v)|^2 dx + \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v) dx
\end{aligned}$$

Note that

$$\begin{aligned}
& \int_M (u-v)(\lambda_u(t)u - \lambda_v(t)v) dx \\
&= (\lambda_u(t) - \lambda_v(t)) \int_M (u-v)u dx + \lambda_v(t) \int_M (u-v)^2 dx \\
&\leq |\lambda_u(t) - \lambda_v(t)| \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \left(\int_M u^2 dx \right)^{\frac{1}{2}} + \lambda_v(t) \int_M (u-v)^2 dx \\
&\leq C |\lambda_u(t) - \lambda_v(t)| \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} + C \int_M (u-v)^2 dx,
\end{aligned}$$

and

$$\begin{aligned}
|\lambda_u(t) - \lambda_v(t)| &= \left| \int_M (|\nabla u|^2 - |\nabla v|^2) + (u^{p+1} - v^{p+1}) dx \right| \\
&\leq \int_M |\nabla(u-v)|(|\nabla u| + |\nabla v|) dx + \left| \int_M (u-v) \left(\frac{u^{p+1} - v^{p+1}}{u-v} \right) dx \right| \\
&\leq \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M (|\nabla u| + |\nabla v|)^2 dx \right)^{\frac{1}{2}} \\
&\quad + \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \left(\int_M \left(\frac{u^{p+1} - v^{p+1}}{u-v} \right)^2 dx \right)^{\frac{1}{2}} \\
(2.19) \quad &\leq C \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} + C \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_M (u-v)^2 dx \\
&\leq - \int_M |\nabla(u-v)|^2 dx + C^2 \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \\
&\quad + C^2 \int_M (u-v)^2 dx + C \int_M (u-v)^2 dx \\
&\leq - \frac{1}{2} \int_M |\nabla(u-v)|^2 dx + \left(\frac{C^4}{2} + C^2 + C \right) \int_M (u-v)^2 dx.
\end{aligned}$$

By Gronwall inequality, we have

$$\|u-v\|_{L^2}^2 \leq \|g_u - g_v\|_{L^2}^2 \exp\left(\left(\frac{C^4}{2} + C^2 + C\right)t\right).$$

Further more,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_M |\nabla(u-v)|^2 dx &= \int_M \nabla(u-v) \cdot \nabla(u-v)_t dx \\
&= - \int_M \Delta(u-v) \cdot (u-v)_t dx \\
&= - \int_M \Delta(u-v) \cdot (\Delta(u-v) + \lambda_u(t)u - \lambda_v(t)v - (u^p - v^p)) dx \\
&= - \int_M (\Delta(u-v))^2 dx + \int_M \nabla(u-v) \cdot \nabla(\lambda_u(t)u - \lambda_v(t)v) dx \\
&\quad + \int_M \Delta(u-v) \cdot (u^p - v^p) dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_M \Delta(u-v) \cdot (u^p - v^p) dx \\
&\leq \frac{1}{2} \int_M (\Delta(u-v))^2 dx + \frac{1}{2} \int_M (u^p - v^p)^2 dx \\
&= \frac{1}{2} \int_M (\Delta(u-v))^2 dx + \frac{1}{2} \int_M (u-v)^2 \left(\frac{u^p - v^p}{u-v} \right)^2 dx \\
&\leq \frac{1}{2} \int_M (\Delta(u-v))^2 dx + \frac{C}{2} \int_M (u-v)^2 dx
\end{aligned}$$

Likewise,

$$\begin{aligned}
& \int_M \nabla(u-v) \cdot \nabla(\lambda_u(t)u - \lambda_v(t)v) dx \\
&= (\lambda_u(t) - \lambda_v(t)) \int_M \nabla(u-v) \cdot \nabla u dx + \lambda_v(t) \int_M |\nabla(u-v)|^2 dx \\
&\leq |\lambda_u(t) - \lambda_v(t)| \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M |\nabla u|^2 dx \right)^{\frac{1}{2}} + \lambda_v(t) \int_M |\nabla(u-v)|^2 dx \\
&\leq C |\lambda_u(t) - \lambda_v(t)| \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} + C \int_M |\nabla(u-v)|^2 dx \\
&\leq C^2 \left(\int_M |\nabla(u-v)|^2 dx + \left(\int_M |\nabla(u-v)|^2 dx \right)^{\frac{1}{2}} \left(\int_M (u-v)^2 dx \right)^{\frac{1}{2}} \right) \\
&\quad + C \int_M |\nabla(u-v)|^2 dx \\
&\leq \left(\frac{3}{2}C^2 + C \right) \int_M |\nabla(u-v)|^2 dx + \frac{C^2}{2} \int_M (u-v)^2 dx,
\end{aligned}$$

where the second inequality follows by (2.19). Then we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_M |\nabla(u-v)|^2 dx &\leq \left(\frac{3}{2}C^2 + C \right) \int_M |\nabla(u-v)|^2 dx + \frac{C^2 + C}{2} \int_M (u-v)^2 dx \\
&\leq C' \int_M |\nabla(u-v)|^2 dx,
\end{aligned}$$

where the second inequality follows by Poincare inequality and C' is a constant depending on the constant of Poincare inequality and C . By Gronwall inequality, we have

$$\|\nabla(u-v)\|_{L^2}^2 \leq \|\nabla(g_u - g_v)\|_{L^2}^2 \exp(C't).$$

□

Finally, we give the proofs of Corollary 5 and Corollary 6 below.

Proof of Corollary 5.

Since

$$\frac{1}{2} \frac{d}{dt} \int_M |\nabla u|^2 dx + \int_M (u_t)^2 dx = \int_M u_t A dx,$$

we have

$$\int_M |\nabla u|^2 dx - \int_M |\nabla g|^2 dx + 2 \int_0^t \int_M (u_t)^2 dx dt = 2 \int_0^t \int_M u_t A dx dt,$$

Since $A \in L^2(\mathbb{R}_+, L^2(M))$, we get

$$(2.20) \quad \int_0^t \int_M (u_t)^2 dx dt \leq \int_M |\nabla g|^2 dx + \int_0^t \int_M A^2 dx dt \leq C.$$

So $\int_s^\infty \int_M (u_t)^2 dx dt \rightarrow 0$ as $s \rightarrow \infty$.

By the arguments in Theorem 1, we have $\lambda(t) = \int_M (|\nabla u|^2 - uA) dx$ is continuous, uniformly bounded in $t \in [0, \infty)$. Moreover, $u \in L^\infty(\mathbb{R}_+, H^1(M))$. Then we can take a subsequence $\{t_i\}$ with $t_i \rightarrow \infty$ such that $u_i(x) = u(x, t_i)$, $\lambda(t_i) \rightarrow \lambda_\infty$. By (2.20) and Theorem 1, we have

$$\begin{cases} u_i \rightarrow u_\infty & \text{in } L^2(M), \\ u_i \rightharpoonup u_\infty & \text{in } H^1(M), \\ \partial_t u_i - (\lambda(t_i) - \lambda_\infty)u_i \rightarrow 0 & \text{in } L^2(M). \end{cases}$$

Since $\partial_t u_i - (\lambda(t_i) - \lambda_\infty)u_i = \Delta u_i + \lambda_\infty u_i + A$, $u_\infty \in H^1$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty + A = 0$ in M and $\int_M |u_\infty|^2 dx = 1$. \square

Proof of Corollary 6.

Since

$$\frac{1}{2} \frac{d}{dt} \int_M |\nabla u|^2 dx = - \int_M (u_t)^2 dx - \frac{1}{p+1} \frac{d}{dt} \int_M u^{p+1} dx,$$

we have

$$(2.21) \quad \lambda(t) + 2 \int_0^t \int_M |u_t|^2 dx dt = \int_M (|\nabla g|^2 + \frac{2}{p+1} g^{p+1}) dx + \frac{p-1}{p+1} \int_M u^{p+1} dx.$$

By the arguments in Theorem 2, we have $\lambda(t) = \int_M (|\nabla u|^2 + u^{p+1}) dx$ is continuous, uniformly bounded in $t \in [0, \infty)$. Moreover, $u \in L^\infty(\mathbb{R}_+, H^1(M))$ and $u \in L^\infty(\mathbb{R}_+, L^{p+1}(M))$. Then we can take a subsequence $\{t_i\}$ with $t_i \rightarrow \infty$ such that $u_i(x) = u(x, t_i)$, $\lambda(t_i) \rightarrow \lambda_\infty$. By (2.21) and Theorem 2, we have

$$\begin{cases} u_i \rightarrow u_\infty & \text{in } L^2(M), \\ u_i \rightharpoonup u_\infty & \text{in } H^1(M) \text{ and } L^p(M), \\ \partial_t u_i - (\lambda(t_i) - \lambda_\infty)u_i \rightarrow 0 & \text{in } L^2(M). \end{cases}$$

Since $\partial_t u_i - (\lambda(t_i) - \lambda_\infty)u_i = \Delta u_i + \lambda_\infty u_i - u_i^p$, $u_i \in H^1$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty - u_\infty^p = 0$ in M and $\int_M |u_\infty|^2 dx = 1$. \square

3. GRADIENT ESTIMATES

This section is devoted to the proofs of Theorem 7 and Theorem 8. We show in this section that the Harnack quantity trick introduced in the fundamental work of P.Li and S.T.Yau. The trick is to find a suitable Harnack quantity and apply the maximum principle in a nice way.

Proof of Theorem 7. Let $u > 0$ be a smooth solution to the heat equation on $M \times [0, T)$

$$(3.1) \quad (\partial_t - \Delta)u = \lambda(t)u + A(x, t).$$

Set

$$w = \log u.$$

Then we have

$$(3.2) \quad (\partial_t - \Delta)w = |\nabla w|^2 + (\lambda + u^{-1}A).$$

Following Li-Yau [15] we let $F = t(|\nabla w|^2 + a(\lambda + u^{-1}A) - aw_t)$ (where $a > 1$) be the Harnack quantity for (3.1). Then we have

$$|\nabla w|^2 = \frac{F}{t} - a(\lambda + u^{-1}A) + aw_t,$$

$$\Delta w = w_t - |\nabla w|^2 - (\lambda + u^{-1}A) = -\frac{F}{at} - (1 - \frac{1}{a})|\nabla w|^2.$$

and

$$w_t - \Delta w = |\nabla w|^2 + (\lambda + u^{-1}A) = \frac{F}{t} + (1 - a)(\lambda + u^{-1}A) + aw_t.$$

Note that

$$(\partial_t - \Delta)w_t = 2\nabla w \nabla w_t + \frac{d}{dt}(\lambda + u^{-1}A).$$

Using the Bochner formula, we have

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla w_t - [2|D^2 w|^2 + 2(\nabla w, \nabla \Delta w) + 2Ric(\nabla w, \nabla w)],$$

and using (3.2) we get

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla(w_t - \Delta w) - [2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)],$$

which can be rewritten as

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla\left[\frac{F}{t} + (1-a)(\lambda + u^{-1}A) + aw_t\right] - [2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)].$$

Then we have

$$\begin{aligned} (\partial_t - \Delta)(|\nabla w|^2 - aw_t) &= 2\nabla w \nabla\left[\frac{F}{t} + (1-a)(\lambda + u^{-1}A)\right] \\ &\quad - [2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)] - a\frac{d}{dt}(\lambda + u^{-1}A). \end{aligned}$$

Hence

$$\begin{aligned} &(\partial_t - \Delta)(|\nabla w|^2 - aw_t + a(\lambda + u^{-1}A)) \\ &= (\partial_t - \Delta)(|\nabla w|^2 - aw_t) + a(\partial_t - \Delta)(\lambda + u^{-1}A) \\ &= 2\nabla w \nabla\left[\frac{F}{t} + (1-a)(\lambda + u^{-1}A)\right] - [2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)] - a\frac{d}{dt}(\lambda + u^{-1}A) \\ &\quad + a(\partial_t - \Delta)(\lambda + u^{-1}A) \\ &= 2\nabla w \nabla\left[\frac{F}{t} + (1-a)(u^{-1}A)\right] - [2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)] - a\Delta(u^{-1}A). \end{aligned}$$

Then we have

$$\begin{aligned} (\partial_t - \Delta)F &= \frac{F}{t} + 2t\nabla w \nabla\left[\frac{F}{t} + (1-a)(u^{-1}A)\right] \\ &\quad - t[2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)] - at\Delta(u^{-1}A). \end{aligned}$$

Assume that

$$\sup_{M \times [0, T]} F > 0.$$

Applying the maximum principle at the maximum point (z, s) , we then have

$$(\partial_t - \Delta)F \geq 0, \quad \nabla F = 0.$$

In the following our computation is always at the point (z, s) . So we get

$$(3.3) \quad \frac{F}{s} + 2(1-a)s\nabla w \nabla(u^{-1}A) - s[2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)] - as\Delta(u^{-1}A) \geq 0.$$

That is

$$(3.4) \quad F - as^2\Delta(u^{-1}A) \geq 2(a-1)s^2\nabla w \nabla(u^{-1}A) + s^2[2|D^2 w|^2 + 2\text{Ric}(\nabla w, \nabla w)].$$

Set

$$\mu = \frac{|\nabla w|^2}{F}(z, s).$$

Then at (z, s) ,

$$|\nabla w|^2 = \mu F.$$

Hence

$$\nabla \frac{A}{u} = \frac{\nabla A}{u} - \frac{A\nabla u}{u^2} = \frac{\nabla A}{u} - \frac{A}{u}\nabla w.$$

So

$$\begin{aligned} \nabla w \cdot \nabla \frac{A}{u} &= \frac{\nabla w \cdot \nabla A}{u} - \frac{A}{u}|\nabla w|^2 \geq -\frac{|\nabla w||\nabla A|}{u} - \frac{A}{u}|\nabla w|^2 \\ (3.5) \quad &= -\frac{|\nabla A|}{u}\sqrt{\mu F} - \frac{A}{u}\mu F \geq -\frac{1}{2}\frac{|\nabla A|^2}{u} - \left(\frac{1}{2} + A\right)\frac{\mu F}{u}. \end{aligned}$$

Further more, we have

$$\begin{aligned}
(\partial_t - \Delta)(u^{-1}A) &= \frac{1}{u}(\partial_t - \Delta)A - \frac{A}{u^2}(\partial_t - \Delta)u + \frac{2}{u^2}\nabla u \cdot \nabla A - 2\frac{A}{u^3}|\nabla u|^2 \\
&= \frac{1}{u}(\partial_t - \Delta)A - \frac{A}{u^2}(\lambda u + A) + \frac{2}{u}\nabla w \cdot \nabla A - 2\frac{A}{u}|\nabla w|^2 \\
&\leq \frac{1}{u}(\partial_t - \Delta)A - \frac{A}{u^2}(\lambda u + A) + \frac{2}{u}\sqrt{\mu F}|\nabla A| - 2\frac{A}{u}\mu F \\
&\leq \frac{1}{u}(\partial_t - \Delta)A - \frac{A}{u}(\lambda + u^{-1}A) + \frac{\mu F}{u} + \frac{|\nabla A|^2}{u} - 2\frac{A}{u}\mu F,
\end{aligned}$$

and

$$\begin{aligned}
\partial_t(u^{-1}A) &= \frac{A_t}{u} - \frac{A}{u^2}u_t \\
&= \frac{A_t}{u} - \frac{A}{u}w_t \\
&= \frac{A_t}{u} - \frac{A}{u}\left(\frac{1}{a}(|\nabla w|^2 - \frac{F}{s}) + (\lambda + u^{-1}A)\right) \\
&= \frac{A_t}{u} - \frac{A}{u} \cdot \frac{F}{a}\left(\mu - \frac{1}{s}\right) - \frac{A}{u}(\lambda + u^{-1}A).
\end{aligned}$$

Hence

$$\begin{aligned}
(3.6) \quad -\Delta(u^{-1}A) &= (\partial_t - \Delta)(u^{-1}A) - \partial_t(u^{-1}A) \\
&\leq -\frac{\Delta A}{u} + \frac{|\nabla A|^2}{u} + \frac{1-2A}{u}\mu F + \frac{A}{u} \cdot \frac{F}{a}\left(\mu - \frac{1}{s}\right) \\
&< -\frac{\Delta A}{u} + \frac{|\nabla A|^2}{u} + \frac{1-2A}{u}\mu F + \frac{A}{u} \cdot \frac{F}{a}\mu.
\end{aligned}$$

Note that

$$|D^2w|^2 + Ric(\nabla w, \nabla w) \geq \frac{1}{n}|\Delta w|^2 - K|\nabla w|^2.$$

So

$$\begin{aligned}
|D^2w|^2 + Ric(\nabla w, \nabla w) &\geq \frac{1}{n}\left(\frac{F}{as} + \left(1 - \frac{1}{a}\right)|\nabla w|^2\right)^2 - K|\nabla w|^2 \\
(3.7) \quad &= \frac{F^2}{n}\left(\frac{1}{as} + \left(1 - \frac{1}{a}\right)\mu\right)^2 - K\mu F.
\end{aligned}$$

Substituting (3.5) (3.6) and (3.7) into (3.4), we get

$$\begin{aligned}
&F + \frac{as^2}{u}(-\Delta A + |\nabla A|^2) + \mu F \frac{s^2}{u}(a + (1-2a)A) \\
&\geq -s^2(a-1)\frac{|\nabla A|^2}{u} - (a-1)s^2\frac{1+2A}{u}\mu F + \frac{2F^2}{n}\left(\frac{1}{a} + \left(1 - \frac{1}{a}\right)\mu s\right)^2 - 2s^2K\mu F.
\end{aligned}$$

Assume that

$$F \geq \frac{as^2}{u}(-\Delta A + |\nabla A|^2) + s^2(a-1)\frac{|\nabla A|^2}{u},$$

for otherwise we are done. Then we have

$$\begin{aligned}
2F + \mu F \frac{s^2}{u}(a + (1-2a)A) + (a-1)s^2\frac{1+2A}{u}\mu F + 2s^2K\mu F \\
\geq \frac{2F^2}{n}\left(\frac{1}{a} + \left(1 - \frac{1}{a}\right)\mu s\right)^2.
\end{aligned}$$

Simplifying this inequality, we get

$$\frac{2F}{n} \frac{1}{a^2} \leq \frac{2}{(1 + (a-1)\mu s)^2} + \frac{\mu s}{(1 + (a-1)\mu s)^2} \cdot s(u^{-1}(a + (1-2a)A) + u^{-1}(a-1)(1+2A) + 2K).$$

Hence we have the estimate for F at (z, s) such that

$$F(z, s) \leq C(u^{-1}, |A|, |\nabla A|, |\Delta A|, K, a, T),$$

which is the desired gradient estimate. \square

The idea proof of Theorem 8 is similar to Theorem 7.

Proof of Theorem 8. Let $u > 0$ be a smooth solution to the heat equation on $M \times [0, T)$

$$(3.8) \quad (\partial_t - \Delta)u = \lambda(t)u - u^p.$$

Set

$$w = \log u.$$

Then we have

$$(3.9) \quad (\partial_t - \Delta)w = |\nabla w|^2 + (\lambda - u^{p-1}).$$

Following Li-Yau [15] we let $F = t(|\nabla w|^2 + a(\lambda - u^{p-1}) - aw_t)$ (where $a > 1$) be the Harnack quantity for (3.8). Then we have

$$|\nabla w|^2 = \frac{F}{t} - a(\lambda - u^{p-1}) + aw_t,$$

$$\Delta w = w_t - |\nabla w|^2 - (\lambda - u^{p-1}) = -\frac{F}{at} - (1 - \frac{1}{a})|\nabla w|^2.$$

and

$$w_t - \Delta w = |\nabla w|^2 + (\lambda - u^{p-1}) = \frac{F}{t} + (1-a)(\lambda - u^{p-1}) + aw_t.$$

Note that

$$(\partial_t - \Delta)w_t = 2\nabla w \nabla w_t + \frac{d}{dt}(\lambda - u^{p-1}).$$

Using the Bochner formula, we have

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla w_t - [2|D^2 w|^2 + 2(\nabla w, \nabla \Delta w) + 2Ric(\nabla w, \nabla w)],$$

and using (3.9) we get

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla (w_t - \Delta w) - [2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)],$$

which can be rewritten as

$$(\partial_t - \Delta)|\nabla w|^2 = 2\nabla w \nabla [\frac{F}{t} + (1-a)(\lambda - u^{p-1}) + aw_t] - [2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)].$$

Then we have

$$\begin{aligned} (\partial_t - \Delta)(|\nabla w|^2 - aw_t) &= 2\nabla w \nabla [\frac{F}{t} + (1-a)(\lambda - u^{p-1})] \\ &\quad - [2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)] - a \frac{d}{dt}(\lambda - u^{p-1}). \end{aligned}$$

Hence

$$\begin{aligned}
& (\partial_t - \Delta)(|\nabla w|^2 - aw_t + a(\lambda - u^{p-1})) \\
= & (\partial_t - \Delta)(|\nabla w|^2 - aw_t) + a(\partial_t - \Delta)(\lambda - u^{p-1}) \\
= & 2\nabla w \nabla \left[\frac{F}{t} + (1-a)(\lambda - u^{p-1}) \right] - [2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)] \\
& - a \frac{d}{dt}(\lambda - u^{p-1}) + a(\partial_t - \Delta)(\lambda - u^{p-1}) \\
= & 2\nabla w \nabla \left[\frac{F}{t} + (a-1)u^{p-1} \right] - [2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)] + a\Delta u^{p-1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
(\partial_t - \Delta)F &= \frac{F}{t} + 2t\nabla w \nabla \left[\frac{F}{t} + (a-1)u^{p-1} \right] \\
&\quad - t[2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)] + at\Delta u^{p-1}.
\end{aligned}$$

Assume that

$$\sup_{M \times [0, T]} F > 0.$$

Applying the maximum principle at the maximum point (z, s) , we then have

$$(\partial_t - \Delta)F \geq 0, \quad \nabla F = 0.$$

In the following our computation is always at the point (z, s) . So we get

$$(3.10) \quad \frac{F}{s} + 2(a-1)s\nabla w \nabla u^{p-1} - s[2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)] + as\Delta u^{p-1} \geq 0.$$

That is

$$(3.11) \quad F + 2(a-1)s^2\nabla w \nabla u^{p-1} + as^2\Delta u^{p-1} \geq s^2[2|D^2 w|^2 + 2Ric(\nabla w, \nabla w)].$$

Set

$$\mu = \frac{|\nabla w|^2}{F}(z, s).$$

Then at (z, s) ,

$$|\nabla w|^2 = \mu F.$$

Hence

$$(3.12) \quad \nabla w \nabla u^{p-1} = (p-1)u^{p-1}|\nabla w|^2 = (p-1)u^{p-1}\mu F.$$

Since

$$\frac{\Delta u}{u} = \frac{u_t}{u} - (\lambda - u^{p-1}) = w_t - (\lambda - u^{p-1}) = \frac{1}{a}(-\frac{F}{s} + |\nabla w|^2).$$

We have

$$\begin{aligned}
\Delta u^{p-1} &= (p-1)u^{p-1}\frac{\Delta u}{u} + (p-1)(p-2)u^{p-3}|\nabla u|^2 \\
&= (p-1)u^{p-1}\frac{1}{a}(-\frac{F}{s} + |\nabla w|^2) + (p-1)(p-2)u^{p-1}|\nabla w|^2 \\
(3.13) \quad &= (p-1)u^{p-1}\frac{1}{a}(-\frac{F}{s} + \mu F) + (p-1)(p-2)u^{p-1}\mu F.
\end{aligned}$$

Note that

$$|D^2 w|^2 + Ric(\nabla w, \nabla w) \geq \frac{1}{n}|\Delta w|^2 - K|\nabla w|^2.$$

So

$$|D^2 w|^2 + Ric(\nabla w, \nabla w) \geq \frac{1}{n}(\frac{F}{as} + (1 - \frac{1}{a})|\nabla w|^2)^2 - K|\nabla w|^2$$

$$(3.14) \quad = \frac{F^2}{n} \left(\frac{1}{as} + \left(1 - \frac{1}{a}\right) \mu \right)^2 - K \mu F.$$

Substituting (3.12) (3.13) and (3.14) into (3.11), we get

$$\begin{aligned} & F + 2s^2(a-1)(p-1)u^{p-1}\mu F + s(p-1)u^{p-1}F(\mu s - 1) \\ & + as^2(p-1)(p-2)u^{p-1}\mu F \geq \frac{2F^2}{n} \left(\frac{1}{a} + \left(1 - \frac{1}{a}\right) \mu s \right)^2 - 2s^2 K \mu F. \end{aligned}$$

Simplifying this inequality, we get

$$\begin{aligned} & \frac{2F}{n} \frac{1}{a^2} \leq \frac{1}{(1 + (a-1)\mu s)^2} + \frac{\mu s}{(1 + (a-1)\mu s)^2} \\ & \cdot s(2(a-1)(p-1)u^{p-1} + (p-1)u^{p-1} + a(p-1)|p-2|u^{p-1} + 2K) \end{aligned}$$

Hence we have the estimate for F at (z, s) such that

$$F(z, s) \leq C(u^{p-1}, K, a, p, T),$$

which is the desired gradient estimate. \square

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